

0-Cohomology of semigroups

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In 1974 my teacher L. M. Gluskin has offered me to study projective representations of semigroups. As one could expect, cohomology appeared in this problem, however it “slightly” differed from the Eilenberg–MacLane cohomology. I have named it “0-cohomology”, have studied its properties insofar as this was necessary for the initial problem, and thought that I will probably never meet this notion again.

However, for the last 30 years I returned to 0-cohomology again and again since I met problems, in which it appeared.

This article is a survey of 0-cohomology. The main attention is devoted to applications. Necessary definitions and results from Homological Algebra and Theory of Semigroups can be found in [5], [7], and [19].

I have to note that there are a lot of various cohomology theories adapted to solution of specific problems in semigroups [6, 12, 18, 23, 24, 25, 42, 44, 46]; see also my review [40]. So an invention of one more cohomology is not any novelty. However, 0-cohomology seems attractive (at least for me!) because it links semigroups with different branches of the algebra.

1 Eilenberg–MacLane cohomology

The definition of semigroup cohomology does not differ from group cohomology [5]: for a semigroup S and a (left) S -module A we call by an n -dimensional cohomology group the group $H^n(S, A) = \text{Ext}_{\mathbb{Z}S}^n(\mathbb{Z}, A)$ where \mathbb{Z} is considered as a trivial $\mathbb{Z}S$ -module. We will name this cohomology by *Eilenberg–MacLane cohomology* or briefly *EM-cohomology*.

In what follows we will use another well-known definition. By $C^n(S, A)$ the group of all n -place mappings $f : \underbrace{S \times \dots \times S}_{n \text{ times}} \rightarrow A$ (the group of n -dimensional cochains) is denoted; a coboundary operator $\partial^n : C^n(S, A) \rightarrow C^{n+1}(S, A)$ is defined as follows:

$$\begin{aligned} \partial^n f(x_1, \dots, x_{n+1}) &= x_1 f(x_2, \dots, x_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i f(x_1, \dots, x_i x_{i+1}, \dots, x_{n+1}) + (-1)^{n+1} f(x_1, \dots, x_n) \end{aligned} \quad (1)$$

Then $\partial^n \partial^{n-1} = 0$, i. e.

$\text{Im} \partial^{n-1} = B^n(S, A)$ (the group of n -dimensional coboundaries)

$\subseteq \text{Ker} \partial^n = Z^n(S, A)$ (the group of n -dimensional cocycles)

and cohomology groups are defined as $H^n(S, A) = Z^n(S, A) / B^n(S, A)$.

However, for cohomology of semigroups one does not manage to obtain results which are comparable to theory of cohomology of groups. In this section we give several results about cohomology of semigroups, illustrating its specifics.

Here is a typical example: since a projective module over a semigroup is not obliged to be projective over its subsemigroup, the lemma of Shapiro [4], which expresses cohomology of a subgroup through cohomology of groups, does not hold for semigroups. So a result, received in [5], is of interest:

Theorem 1 *Let T be a subsemigroup of a group G . If G is a group of fractions for T (i. e. each element from G can be written in the form $x^{-1}y$ for some $x, y \in T$), then homomorphisms $i^n : H^n(G, A) \rightarrow H^n(T, A)$, induced by the embedding $i : T \rightarrow G$, are bijective for any G -module A .*

Let I be an ideal of a semigroup S . What can one say about homomorphisms $\varepsilon^n : H^n(S, A) \rightarrow H^n(I, A)$, induced by the embedding $\varepsilon : I \rightarrow S$? It is easy to show that ε^0 is always an isomorphism and ε^1 is a monomorphism. Using technique of adjoint functors, Adams and Rieffel [1] proved

Theorem 2 *Let I be a left ideal of a semigroup S , having an identity e . Then for any S -module A and for any $n \geq 0$*

$$H^n(S, A) \cong H^n(I, A) \cong H^n(I, eA).$$

In particular, if S contains zero then $H^n(S, A) = 0$ for $n > 0$.

In [1] by help of Theorem 2 a sufficient condition was obtained for an associative algebra over \mathbb{R} be a semigroup algebra.

The connection between $H^n(S, A)$ and $H^n(I, A)$ becomes more close when we take for I the so-called Sushkevich kernel (the least two-sided ideal). This situation was in detail considered by W. Nico [27]. I will not formulate this result, but only note that it implies the description of cohomology of completely simple semigroups:

Theorem 3 *Let S be a completely simple semigroup, G its basic group, e the identity of G , A an S -module. Then $H^n(S, A) \cong H^n(G, eA)$ for any $n \geq 3$.*

A number of papers is devoted to study of cohomological dimension of semigroups. *Cohomological dimension* ($\text{cd}S$) of a semigroup S is defined as the least integer n such that $H^{n+1}(S, A) = 0$ for any S -module A .

It is well-known [5] that cohomological dimension of both a free group and a free semigroup (monoid) does not exceed 1. In the case of groups the converse statement is also true (this is the famous Stallings–Swan theorem [4]). For semigroups the situation is evidently different: joining an extra zero to any semigroup makes trivial all its cohomology groups, except 0-dimensional. So it is naturally to confine ourselves to the class of semigroups with cancellation. Mitchell [26] has shown that the free product of a free group and a free monoid (which he called a partially free monoid) has cohomological dimension 1. In the same paper he has formulated a suggestion: if S is a cancellative monoid with $\text{cd}S = 1$ then S is partially free.

In [32] (and later in [34] more generally) a counter-example to the Mitchell suggestion was built and a “weakened Mitchell conjecture” was proposed: if S is a cancellative monoid with $\text{cd}S = 1$ then S is embedded into a free group. This suggestion was proved in [39]. Probably, it is difficult to get more exact information: it is shown in [35] that a semigroup antiisomorphic to the counter-example from [32] (and therefore also embeddable into a free group) has cohomological dimension 2. A good answer is known only in the commutative case: cohomological dimensions of all subsemigroups of \mathbb{Z} are 1 [36].

2 Properties of 0-cohomology

Before constructing 0-cohomology we have to define a suitable Abelian category. In what follows S is a semigroup with a zero.

Definition 1 *A 0-module over S is an Abelian (additive) group A equipped with a multiplication $(S \setminus 0) \times A \rightarrow A$ satisfying the following conditions for all $s, t \in S \setminus 0$, $a, b \in A$:*

$$s(a + b) = sa + sb,$$

$$st \neq 0 \Rightarrow s(ta) = (st)a.$$

A morphism of 0-modules is a homomorphism of Abelian groups $\varphi : A \rightarrow B$ such that $\varphi(sa) = s\varphi(a)$ for $s \in S \setminus 0$, $a \in A$.

We will denote the obtained category of 0-modules by $\text{Mod}_0 S$. It is easy to see that for the semigroup $T^0 = T \cup 0$ with an extra zero the category $\text{Mod}_0 T^0$ is isomorphic to the category $\text{Mod} T$ of usual modules over T . It turns out that in general case $\text{Mod}_0 S$ is also isomorphic to a category of (usual) modules over some semigroup.

Denote by \overline{S} the set of all finite sequences (x_1, \dots, x_m) such that $x_i \in S \setminus 0$ ($1 \leq i \leq m$) and $x_i x_{i+1} = 0$ ($1 \leq i < m$); thus, all one-element sequences, except (0) , contain in \overline{S} . Define a binary relation ρ on \overline{S} via $(x_1, \dots, x_m) \rho (y_1, \dots, y_n)$ if and only if one of the following conditions is fulfilled:

- 1) $m = n$ and there exists i ($1 \leq i \leq m - 1$) that $x_i = y_i u$, $y_{i+1} = u x_{i+1}$ for some $u \in S$, and $x_j = y_j$ for $j \neq i, j \neq i + 1$;
- 2) $m = n + 1$ and there exists i ($2 \leq i \leq m - 1$) that $x_i = uv$, $y_{i-1} = x_{i-1} u$, $y_i = v x_{i+1}$ for some $u, v \in S$, $x_j = y_j$ for $1 \leq j \leq i - 2$, and $x_j = y_{j-1}$ for $i + 2 \leq j \leq m$.

Let $\overline{\rho}$ be the least equivalence containing ρ , \tilde{S} the quotient set $\overline{S}/\overline{\rho}$. The image of $(x_1, \dots, x_m) \in \overline{S}$ in \tilde{S} will be denoted by $[x_1, \dots, x_m]$.

Define a multiplication on \tilde{S} :

$$[x_1, \dots, x_m][y_1, \dots, y_n] = \begin{cases} [x_1, \dots, x_m y_1, \dots, y_n], & \text{if } x_m y_1 \neq 0, \\ [x_1, \dots, x_m, y_1, \dots, y_n], & \text{if } x_m y_1 = 0. \end{cases}$$

Then \tilde{S} becomes a semigroup, which is called a *gown* of S .

Each 0-module over S can be transformed into an (usual) module over \tilde{S} by

$$[x_1, \dots, x_m]a = x_1(\dots(x_ma)\dots)$$

for $x_1, \dots, x_m \in S \setminus 0$, $a \in A$. Hence we obtain

Proposition 1 $\text{Mod}_0 S \cong \text{Mod } \tilde{S}$.

Corollary 1 *The category $\text{Mod}_0 S$ is Abelian.*

Here are some simplest properties of the gown [33]:

1. If $S = T^0$ is a semigroup with an extra zero then $\tilde{S} \cong S \setminus 0 = T$.
2. It follows from the definition of the relation ρ that the map $S \setminus 0 \rightarrow \tilde{S}$, $x \rightarrow [x]$, is bijective.
3. The subset $J = \{[x_1, \dots, x_m] \in \tilde{S} \mid m > 1\}$ is an ideal in \tilde{S} and $\tilde{S}/J \cong S$.

It is easy to find the gown if the semigroup S is given by defining relations. We will write $S = \langle a_1, \dots, a_m \mid P_i = Q_i, 1 \leq i \leq n \rangle$ if S is generated by elements a_1, \dots, a_m and is defined by equalities $P_1 = Q_1, \dots, P_n = Q_n$. If the value of a word P_i (or the same, of Q_i) in semigroup S is 0 then the equality $P_i = Q_i$ is called *zero*.

Proposition 2 *Let $S = \langle a_1, \dots, a_m \mid P_i = Q_i, 1 \leq i \leq n \rangle$ be a semigroup with a zero, in which none of generating elements is 0. If one deletes all zero defining relations then the obtained semigroup will be isomorphic to the gown \tilde{S} .*

Example 1 *If S is a semigroup with zero multiplication ($S^2 = 0$) then \tilde{S} is a free semigroup.*

Let now A be a 0-module over S .

Definition 2 *A partial n -place mapping from S into A , defined on all n -tuples (x_1, \dots, x_n) such that $x_1 \cdots x_n \neq 0$, is called a n -dimensional cochain. The group of n -dimensional cochains is denoted by $C_0^n(S, A)$. A coboundary operator $\partial^n : C_0^n(S, A) \rightarrow C_0^{n+1}(S, A)$ is given as above, by the formula (1). Then $\partial^n \partial^{n-1} = 0$ and 0-cohomology groups are defined as $H_0^n(S, A) = Z_0^n(S, A) / B_0^n(S, A)$, where $Z_0^n(S, A) = \text{Ker } \partial^n$ is the group of n -dimensional 0-cocycles, $B_0^n(S, A) = \text{Im } \partial^{n-1}$ is the group of n -dimensional 0-coboundaries.*

Example 2 Let $S = T^0 = T \cup 0$ be a semigroup with an extra zero. Then $\tilde{S} \cong T$ and one can easily check that $H_0^n(S, A) \cong H^n(T, A)^1$ while $H^n(S, A) = 0$.

This example shows that 0-cohomology is a generalization of EM-cohomology. In view of Proposition 1, in general case it is naturally to compare the groups $H_0^n(S, A)$ and $H^n(\tilde{S}, A)$. We describe this comparison in more details.

Since the sequence of functors $\{H_0^n(S, -)\}_{n \geq 0}$ is connected by terminology of [19] and $\{H^n(\tilde{S}, -)\}_{n \geq 0}$ is a sequence of derived functors in the isomorphic categories $\text{Mod}_0 S$ and $\text{Mod } \tilde{S}$ respectively, then an isomorphism

$$\varepsilon^0 : H^0(\tilde{S}, A) = \text{Hom}_{\text{Mod } \tilde{S}}(\mathbb{Z}, A) \cong \text{Hom}_{\text{Mod}_0 S}(\mathbb{Z}, A) = H_0^0(S, A)$$

induces group homomorphisms

$$\varepsilon^n : H^n(\tilde{S}, A) \rightarrow H_0^n(S, A)$$

such that $\{\varepsilon^n\}_{n \geq 0}$ are morphisms of cohomology functors.

The homomorphism ε^n is described as follows. If $f \in C^n(\tilde{S}, A)$ then set $(\eta^n f)(x_1, \dots, x_n) = f([x_1], \dots, [x_n])$ for $x_1 \cdot \dots \cdot x_n \neq 0$. Then η^n is a homomorphism from $C^n(\tilde{S}, A)$ into $C_0^n(S, A)$ and it induces the homomorphism ε^n .

Direct calculations prove

Theorem 4 ε^1 is an isomorphism for any semigroup S .

Using the corresponding long exact sequence, from Theorem 4 we obtain

Corollary 2 ε^2 is a monomorphism for any semigroup S .

Generally speaking, the groups $H_0^2(S, A)$ and $H^2(\tilde{S}, A)$ are not isomorphic:

Example 3 Let a commutative semigroup S consists of elements $u, v, w, 0$ with the multiplication

$$u^2 = v^2 = uv = w, \quad uw = vw = 0.$$

One can show that $H^2(\tilde{S}, A) = 0$ for any module A over S . On the other hand, if A (considered now as a 0-module) is not one-element and $a \in A \setminus 0$, then 0-cocycle f defined by the condition

$$f(x, y) = \begin{cases} a & \text{for } x = y = u, \\ 0 & \text{otherwise,} \end{cases}$$

is not a 0-coboundary and thus $H_0^2(S, A) \neq 0$.

¹Here we consider A both as a 0-module over S and a module over \tilde{S} , which does not lead to misunderstanding in this context.

Apropos, this example shows that 0-cohomology is not a derived functor unlike EM-cohomology. Indeed, according to Proposition 1 there are injective objects in the category $\text{Mod}_0 S$ and a derived functor must vanish on them.

In this situation semigroups categorical at zero are of special interest. We recall that a semigroup S with a zero is called *categorical at zero* if for any $x, y, z \in S$ from $xyz = 0$ it follows either $xy = 0$ or $yz = 0$. For instance, if we join a new element 0 to the set of all morphisms of a small category and set product of morphisms equal to 0 when their composition is not defined, then the obtained set becomes a semigroups categorical at zero.

Theorem 5 [28] *If semigroup S is categorical at zero then ε^n is an isomorphism for any $n \geq 0$.*

This theorem is used in two ways. On the one hand, since, for instance, completely 0-simple semigroups are categorical at zero then by Theorem 5 one succeeds to calculate their 0-cohomology, reducing this problem to EM-cohomology [30]. On the other hand, in concrete examples usually it is easier to calculate 0-cohomology of a given semigroup, and then use it for finding EM-cohomology of its gown. In more detail we will consider this question in the following section.

Example 3 shows that in general Theorem 5 is not satisfied.

3 Calculating EM-cohomology

The free product of semigroups gives a first example of the using 0-cohomology:

Theorem 6 *Let S, T be semigroups, $S * T$ their free product. Then*

$$H^n(S * T, A) \cong H^n(S, A) \oplus H^n(T, A)$$

*for any $n \geq 2$ and for any $(S * T)$ -module A .*

Proof. 0-Direct union $S^0 \sqcup_0 T^0$ is a semigroup categorical at zero. Besides, $\widetilde{S^0 \sqcup_0 T^0} \cong S * T$. So

$$\begin{aligned} H^n(S * T, A) &\cong H^n(S^0 \sqcup_0 T^0, A) \cong H^n(S^0, A) \oplus H^n(T^0, A) \\ &\cong H^n(S, A) \oplus H^n(T, A). \end{aligned}$$

Here, the first isomorphism follows from Theorem 5, second is checked directly, and third follows from Example 2.

Remark. There is an analogue of Theorem 6 for groups, but its proof is more complicated.

In Section 1 the notion of cohomological dimension was already mentioned. A counter-example to Mitchell conjecture was obtained by using 0-cohomology. This is the semigroup

$$S = \langle a, b, c, d \mid ab = cd \rangle.$$

Its subset $I = S \setminus \{a, b, c, d, ab\}$ is an ideal and $\widetilde{S/I} \cong S$. By Corollary 2 $H^2(S, A)$ is embedded into $H_0^2(S/I, A)$.

On the other hand, it is easy to see that $f = \partial\varphi$ for any 0-cocycle $f \in Z_0^2(S/I, A)$ if we set

$$\varphi(u) = \begin{cases} f(a, b), & \text{if } u = a, \\ f(c, d), & \text{if } u = c, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $H^2(S, A) = H_0^2(S/I, A) = 0$.

I give one more example of calculation of EM-cohomology for a pair of anti-isomorphic semigroups.

Let $p, q \in \mathbb{N}$. Consider the semigroup

$$T = \langle a, b \mid ab = b, a^p = a^q \rangle.$$

It is the gown for the semigroup

$$S = \langle x, y \mid xy = y, x^p = x^q, yx = y^2 = 0 \rangle.$$

The last is categorical at zero and consists of elements x^k ($k > 0$), y and 0. Its 0-cohomology is easily computed directly, and thus we get:

Proposition 3 [31] *For any T -module A*

- 1) $H^1(T, A) = H_0^1(S, A) \cong A/\{m \in A \mid am = 2m\};$
- 2) $H^n(T, A) = H_0^n(S, A) = 0$ for $n \geq 0$.

For the semigroup T^{op} anti-isomorphic to T , i. e.

$$T^{\text{op}} = \langle a, b \mid ba = b, a^p = a^q \rangle,$$

the answer is more complicated:

Proposition 4 [31] *Let A be an arbitrary T^{op} -module, A_1 its additive group, considered as a T^{op} -module with trivial multiplication, $\langle a \rangle$ the subsemigroup*

generated by the element a . Let homomorphisms ψ^n be induced by embedding $\langle a \rangle \rightarrow T^{\text{op}}$. Then the sequence

$$\begin{aligned} 0 \rightarrow H^0(T^{\text{op}}, A) \xrightarrow{\psi^0} H^0(\langle a \rangle, A) \rightarrow H^0(\langle a \rangle, A_1) \rightarrow H^1(T^{\text{op}}, A) \xrightarrow{\psi^1} \dots \\ \dots \rightarrow H^n(T^{\text{op}}, A) \xrightarrow{\psi^n} H^n(\langle a \rangle, A) \rightarrow H^n(\langle a \rangle, A_1) \rightarrow \dots \end{aligned}$$

is exact.

In particular, if A_1 is torsion-free then $H^n(T^{\text{op}}, A) \cong H^n(\langle a \rangle, A)$ for $n \geq 1$.

More general results in calculation of EM-cohomology were obtained by partial cohomology (see Sec. 8).

Finally let me formulate an unsolved problem. Theorem 2 shows that in some cases the cohomology of a semigroup is defined by the cohomology of its ideal. Generally speaking, this is not the case. When I is a two-sided ideal, it is desirable to use for calculation of $H^n(S, A)$ not only the cohomology of the ideal, but also of the quotient semigroup S/I . However, EM-cohomology of the latter is always trivial, because S/I contains the zero element. So a question arises: how the group $H^n(S, A)$ depends on $H^n(I, A)$ and $H_0^n(S/I, A)$ (as well as, maybe, on the cohomology groups of smaller dimension)?

4 Projective representations

In this section we use the following notation: S is an arbitrary semigroup (for simplicity we suppose that it contains an identity), K is a field, K^\times its multiplicative group, $\text{Mat}_n K$ the multiplicative semigroup of all $(n \times n)$ -matrices over K (we will often delete the subscript n). Define an equivalence λ on $\text{Mat}_n K$: for $A, B \in \text{Mat}_n K$ put

$$A \lambda B \iff \exists c \in K^\times : A = cB.$$

Evidently λ is a congruence of $\text{Mat}_n K$. The quotient semigroup $\text{PMat}_n K = \text{Mat}_n K / \lambda$ will be called *a semigroup of projective $(n \times n)$ -matrices*.

Let $\Delta : S \rightarrow \text{PMat}_n K$ be a homomorphism and $\alpha : \text{Mat}_n K \rightarrow \text{PMat}_n K$ be the canonical homomorphism, corresponding to congruence λ . We fix a mapping $\beta : \text{PMat}_n K \rightarrow \text{Mat}_n K$, choosing representatives in λ -classes. If we denote $\Gamma = \beta\Delta$ then $\Delta = \alpha\beta\Delta = \alpha\Gamma$. Since Δ and α are homomorphisms,

$$\alpha\Gamma(xy) = \Delta(x)\Delta(y) = \alpha(\Gamma(x)\Gamma(y))$$

for all $x, y \in S$. Hence $\Gamma(xy)$ and $\Gamma(x)\Gamma(y)$ vanish simultaneously. So we come to the following definition:

Definition 3 *The mapping $\Gamma : S \rightarrow \text{Mat}_n K$ is called a projective representation² of S over K if it satisfies the following conditions:*

- 1) $\Gamma(xy) = 0 \iff \Gamma(x)\Gamma(y) = 0$ for all $x, y \in S$;
- 2) *there is a partially defined mapping $\rho : S \times S \rightarrow K^\times$ such that*

$$\text{dom } \rho = \{(x, y) \mid \Gamma(xy) \neq 0\} \quad (2)$$

and

$$\forall (x, y) \in \text{dom } \rho \quad \Gamma(x)\Gamma(y) = \Gamma(xy)\rho(x, y). \quad (3)$$

The mapping ρ is called a factor set of Γ and the number n the degree of Γ .

Remark. It is easy to see that (3) remains valid for all $x, y \in S$ if we extend ρ to a completely defined mapping setting $\rho(x, y) = 0$ for x, y such that $\rho(x, y)$ was not defined. Hereinafter we will often suppose this.

As in the case of projective representations of groups, it is desirable to have independent characterization of partially defined mappings $\rho : S \times S \rightarrow K^\times$ which can serve as factor sets for some projective representations of S . Applying (3) to the equality $\Gamma(x)[\Gamma(y)\Gamma(z)] = [\Gamma(x)\Gamma(y)]\Gamma(z)$, we get:

$$\rho(x, y)\rho(xy, z) = \rho(x, yz)\rho(y, z) \quad (4)$$

for all $x, y, z \in S$. However, unlike projective representations of groups, condition (4) is not sufficient.

Theorem 7 [29] *A mapping $\rho : S \times S \rightarrow K$ is a factor set for a certain (possible, infinite-dimensional) projective representation of a monoid S if and only if ρ satisfies (4) and for all $x, y \in S$*

$$\rho(x, y) = 0 \iff \rho(1, xy) = 0. \quad (5)$$

Similarly to groups, the choice of different representatives of the λ -classes leads to an equivalent projective representation. So we call factor sets ρ and σ *equivalent* ($\rho \sim \sigma$) if they vanish simultaneously and there exists a function $\alpha : S \rightarrow K^\times$ such that for all $x, y \in S$ we have

$$\rho(x, y) = \alpha(x)\alpha(xy)^{-1}\alpha(y)\sigma(x, y).$$

Define the product of factor sets ρ and σ by pointwise multiplication: $\rho\sigma(x, y) = \rho(x, y)\sigma(x, y)$. It follows immediately from Theorem 7 that $\rho\sigma$ is also a factor set. So the set $m(S)$ of all factor sets is a semigroup and \sim is

²One also calls the homomorphism Δ by a projective representation.

its congruence. The quotient semigroup $M(S) = m(S)/\sim$ is called a *Schur multiplier* of S .

For groups the Schur multiplier is isomorphic to the group $H^2(G, K^\times)$ [8]. In our situation it is a commutative inverse semigroup. Consider the construction of the semigroups $M(S)$ and $m(S)$.

Since $m(S)$ and $M(S)$ are commutative, it follows from Clifford theorem [7] that they are strong semilattices of groups:

$$m(S) = \bigcup_{\alpha \in b(S)} m_\alpha(S), \quad M(S) = \bigcup_{\alpha \in B(S)} M_\alpha(S),$$

where $b(S)$ and $B(S)$ are semilattices, $m_\alpha(S)$ and $M_\alpha(S)$ are groups. We will call $m_\alpha(S)$ and $M_\alpha(S)$ *components* of semigroups $m(S)$ and $M(S)$ respectively.

The first step in our consideration is a description of idempotent factor sets:

Lemma 1 *There is a bijection $\varepsilon \leftrightarrow I_\varepsilon$ between idempotents $\varepsilon \in m(S)$ and ideals of S such that*

$$\varepsilon(x, y) = \begin{cases} 1, & \text{if } xy \notin I_\varepsilon, \\ 0, & \text{if } xy \in I_\varepsilon, \end{cases}$$

and

$$I_{\varepsilon_1 \varepsilon_2} = I_{\varepsilon_1} \cup I_{\varepsilon_2}. \quad (6)$$

We will denote by $Y(S)$ the semilattice of all (two-sided) ideals of S with respect to union. We consider the empty subset as an ideal too, i. e. $\emptyset \in Y(S)$.

Corollary 3 $b(S) \cong B(S) \cong Y(S)$.

It follows that the ideals $I \in Y(S)$ can serve as indexes for components of the semigroups $m(S)$ and $M(S)$; thus

$$m_I(S)m_J(S) \subseteq m_{I \cup J}(S), \quad M_I(S)M_J(S) \subseteq M_{I \cup J}(S).$$

Let ε_I be the identity of the group $m_I(S)$. Then

$$\varepsilon_I(x, y) = 0 \iff xy \in I.$$

Lemma 2 *The group $m_I(S)$ consists of factor sets ρ for which*

$$\rho(x, y) = 0 \iff xy \in I.$$

Hence groups $m_I(S)$ and $m_0(S/I)$ are isomorphic for $I \neq \emptyset$. If $I = \emptyset$ we have $m_\emptyset(S) \cong m_0(S^0)$. Certainly, this holds for the multiplier too:

Corollary 4 $M_I(S) \cong M_0(S/I)$ if $I \neq \emptyset$, and $M_\emptyset(S) \cong M_0(S^0)$.

Finally, it is easy to see that $M_0(S) \cong H_0^2(S, K^\times)$, and we get the final result:

Theorem 8 [29] *The Schur multiplier $M(S, K)$ of a semigroup S over a field K is isomorphic to the semilattice Y of Abelian groups $H_0^2(S/I, K^\times)$, where $I \in Y$, and K^\times is considered as a trivial 0-module over S/I .*

Further description of projective representations of semigroups was carried out in [29]; it is similar to description of linear representations [7].

5 Brauer monoid

In several articles Haile, Larson and Sweedler [14, 15, 16, 45], see also [17], studied so called strongly primary algebras. Their definition is rather bulky and we will not need it. Instead of this I cite their description, given in [15].

Let K/L be a finite Galois extension with the Galois group G . A *weak 2-cocycle* [45] is defined as a mapping $f : G \times G \rightarrow K$ such that for any $\sigma, \tau, \omega \in G$

$$\begin{aligned}\sigma[f(\tau, \omega)]f(\sigma\tau, \omega) &= f(\sigma, \tau)f(\sigma\tau, \omega), \\ f(1, \sigma) &= f(\sigma, 1) = 1\end{aligned}$$

(hence weak 2-cocycles can take zero value unlike usual cocycles).

Let f be a weak 2-cocycle. On the set A of formal sums of the form $\sum_{\sigma \in G} a_\sigma \sigma$, $a_\sigma \in K$, we define a multiplication by the rule:

$$a\sigma \cdot b\tau = a\sigma(b)f(\sigma, \tau)\sigma\tau, \quad \sigma, \tau \in G, \quad a, b \in K.$$

Then A becomes an associative algebra. The class of such algebras coincides with class of strongly primary algebras.

Strongly primary algebras give a generalization of central simple algebras. In accordance with this Haile, Larson and Sweedler introduced a notion of a Brauer monoid as generalization of a Brauer group. For this aim an equivalence of weak 2-cocycles is defined: $f \sim g$ if there exists a mapping $p : G \rightarrow K^\times$ such that

$$g(\sigma, \tau) = f(\sigma, \tau)p(\sigma)p(\tau)(p(\sigma\tau))^{-1}$$

for any $\sigma, \tau \in G$ under condition $f(\sigma, \tau) \neq 0$. After factorization by this equivalence the set of weak 2-cocycles turns into the *Brauer monoid* $Br(G, K)$ which is an inverse semigroup like the Schur multiplier from Sec. 4. More exactly, denoting by E the semilattice of all idempotents from $Br(G, K)$ (i. e. weak cocycles taking only values 0 and 1), we obtain:

Theorem 9 [16] *$Br(G, K)$ is a semilattice E of Abelian groups $Br_e(G, K)$, where $e \in E$ and $Br_e(G, K)$ consists of all weak 2-cocycles which vanish simultaneously with e . In particular, if $e \equiv 1$ then $Br_e(G, K) \cong H^2(G, K^\times)$ is the Brauer group.*

It turns out [20, 37] that this construction is reduced to 0-cohomology. Let $e \in E$. Join an extra zero 0 to G and define a new operation on G^0 :

$$x \circ y = \begin{cases} xy, & \text{if } e(x, y) = 1, \\ 0, & \text{if } e(x, y) = 0 \end{cases}$$

and besides, $x \circ 0 = 0 \circ x = 0$. With this operation G^0 is a semigroup which we will denote by G_e . Conversely, call by a *modification* $G(\circ)$ of the group G a monoid on G^0 with an operation \circ such that $x \circ y$ is either xy or 0, and moreover $0 \circ x = x \circ 0 = 0$. It is easy to see that there is a bijective correspondence between idempotent weak 2-cocycles and modifications of G . The group K^\times turns into a 0-module over G_e , $Br_e(G, K) \cong H_0^2(G_e, K^\times)$ and Theorem 9 changes into the following statement:

Theorem 10 *$Br(G, K)$ is a semilattice of Abelian groups $H_0^2(G(\circ), K^\times)$, where $G(\circ)$ runs the set of all modifications of the group G .*

It is shown in [37] that in this problem 0-cohomology is used essentially: to describe some properties of Brauer monoid one has to use 0-cohomology of other (different from modifications) semigroups.

Thus study of the Brauer monoid is reduced to description of modifications of the group and their 0-cohomology. However, it is necessary to note that the study of modifications is a difficult combinatorial problem. In general case for a finite group G (only such groups are considered in Haile–Larson–Sweedler theory) we can only confirm that each modification is an union of the subgroup of its invertible elements and a nilpotent ideal. Besides, modifications are 0-cancellative (if $ax = bx \neq 0$ or $xa = xb \neq 0$ then $a = b$). Some examples of modification were considered in [38] and [41].

6 Partial representations of groups

Results of this section were received when I worked in São Paulo, Brasil, thanks to the foundation FAPESR. They were announced on XVIII Brazilian Algebra Meeting [9] and are preparing for publication.

In connection with studying C^* -algebras so called partial linear representations of groups appeared [10, 43]. It is naturally to ask: how do partial **projective** representations of groups look like? It turned out that here 0-cohomology appears too. We start with necessary definitions from [10].

Definition 4 *A mapping $\varphi : G \rightarrow S$ from a group G into a semigroup S is called a partial homomorphism if for all $x, y \in G$*

$$\begin{aligned}\varphi(x^{-1})\varphi(x)\varphi(y) &= \varphi(x^{-1})\varphi(xy) \\ \varphi(x)\varphi(y)\varphi(y^{-1}) &= \varphi(xy)\varphi(y^{-1}) \\ \varphi(x)\varphi(e) &= \varphi(x)\end{aligned}$$

(these equalities imply $\varphi(e)\varphi(x) = \varphi(x)$).

In particular, a partial linear representation (PLR) over a field K is a partial homomorphism into the matrix semigroup, $\Delta : G \rightarrow \text{Mat}_n K$.

R. Exel introduced a monoid $\Sigma(G)$ which plays a special role here. It is generated by symbols $[x]$ ($x \in G$) with defining relations

$$\begin{aligned}[x^{-1}][x][y] &= [x^{-1}][xy] \\ [x][y][y^{-1}] &= [xy][y^{-1}] \\ [x][e] &= [x]\end{aligned}$$

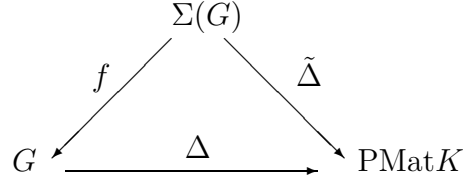
(these equalities imply $[e][x] = [x]$).

$\Sigma(G)$ possesses the following universal property:

1. The mapping $f : G \rightarrow \Sigma(G)$, $f(x) = [x]$, is a partial homomorphism.
2. For any semigroup S and any partial homomorphism $\varphi : G \rightarrow S$ there exists an unique (usual) homomorphism $\tilde{\varphi} : \Sigma(G) \rightarrow S$ such that $\varphi = \tilde{\varphi}f$.

Due to this property the study of PLR's of groups is equivalent to the study of linear representations of its Exel monoid.

It is naturally to define (and to study) partial projective representations of groups by means of usual projective representations of $\Sigma(G)$: we will call the partial homomorphism $\Delta : G \rightarrow \text{PMat}K$ by a *partial projective representation (PPR)* of G (cf. the footnote on p. 10). We get the diagram



where $\tilde{\Delta}$ is a projective representation of $\Sigma(G)$.

However we will see below that PPR's are not reduced to projective representations of semigroups unlike linear ones.

The first step in study of PPR's is a translation of their definition into the language of usual matrices:

Theorem 11 *A mapping $\Gamma : G \rightarrow \text{Mat } K$ is PPR of G if and only if the following conditions hold:*

1) for all $x, y \in G$

$$\Gamma(x^{-1})\Gamma(xy) = 0 \iff \Gamma(x)\Gamma(y) = 0 \iff \Gamma(xy)\Gamma(y^{-1}) = 0;$$

2) there is a mapping $\sigma : G \times G \rightarrow K$ such that

$$\Gamma(x)\Gamma(y) = 0 \iff \sigma(x, y) = 0$$

and

$$\begin{aligned}
\Gamma(x^{-1})\Gamma(x)\Gamma(y) &= \Gamma(x^{-1})\Gamma(xy)\sigma(x, y), \\
\Gamma(x)\Gamma(y)\Gamma(y^{-1}) &= \Gamma(xy)\Gamma(y^{-1})\sigma(x, y).
\end{aligned}$$

Note that this theorem gives another definition of PPR independent of $\Sigma(G)$.

We call σ a *factor set* of Γ and define a product of factor sets as above. However, it is not evident that this product is also a factor set. To prove this, one have to use the Exel monoid again:

Proposition 5 *Let $\sigma : G \times G \rightarrow K$ be a mapping for which there is a factor set ρ of the semigroup $\Sigma(G)$, such that:*

- 1) $\forall x, y \in G \quad \sigma(x, y) = 0 \iff \rho([x], [y]) = 0;$
- 2) $\sigma(x, y) \neq 0 \implies \sigma(x, y) = \frac{\rho([x], [y]) \rho([x^{-1}], [x][y])}{\rho([x^{-1}], [xy])}.$

Then σ is a factor set of some PPR of G .

The converse also holds.

Now the desired result about product comes directly. Moreover:

Corollary 5 *The factor sets of G form a commutative inverse semigroup $Pm(G)$.*

Again, as in Sec.4, we define an equivalence of factor sets and call the respective quotient semigroup by *the Schur multiplier* $PM(G)$. The Schur multiplier is also a commutative inverse semigroup. However, the Schur multipliers of G and of $\Sigma(G)$ are different: one can only confirm that $PM(G)$ is an image of $M(\Sigma(G))$ (Prop.5).

At present the main problem in description of a Schur multiplier is to find such a definition of factor sets which would be independent on both PPR and $\Sigma(G)$. We recall that for usual projective representations of groups such a definition is the cohomological equation (4). In the case of semigroups (more exactly, monoids) it is necessary to add the condition (5). Unfortunately, a cohomological equation is not true for PPR's. At most we can assert

Proposition 6 *Let Γ be a PPR with a factor set σ . Then*

$$\forall x, y, z \in G \quad \Gamma(x)\Gamma(y)\Gamma(z) \neq 0 \implies \sigma(x, y)\sigma(xy, z) = \sigma(x, yz)\sigma(y, z).$$

Later I will give an example where the cohomological equation does not hold.

Now we consider the structure of a Schur multiplier. Certainly, it is a semilattice of their subgroups. So first of all it is necessary to describe its idempotents:

Theorem 12 *Let $\sigma : G \times G \rightarrow K$ be a mapping taking only values 0 and 1 and $\sigma(1, 1) = 1$. Then σ is a factor set if and only if*

$$\forall x, y \in G \quad \sigma(x, y) = 1 \implies \sigma(xy, y^{-1}) = \sigma(y^{-1}, x^{-1}) = \sigma(x, 1) = 1. \quad (7)$$

Example 4 *Let $G = \langle a, b, c \rangle$ be an elementary Abelian group of order 8 with generators a, b, c . Denote $H = \langle b, c \rangle$, $F = (H \setminus 1) \times (H \setminus 1) \setminus \nabla$ where ∇ is a diagonal of the Cartesian square $H \times H$. Set*

$$\sigma(x, y) = \begin{cases} 1 & \text{if } (x, y) \notin F, \\ 0 & \text{if } (x, y) \in F. \end{cases}$$

It is easy to check by Theorem 12 that σ is a factor set. However,

$$\sigma(b, a)\sigma(ba, ac) = 1 \neq 0 = \sigma(b, c)\sigma(a, ac)$$

and the cohomological equation does not hold for $x = b$, $y = a$, $z = ac$.

One can place Theorem 12 into a more general form. Consider an abstract semigroup \mathcal{T} generated by elements α, β, γ with defining relations

$$\left. \begin{aligned} \alpha^2 = \beta^2 = 1, \quad (\alpha\beta)^2 = 1 \\ \gamma^2 = 1, \quad \alpha\gamma = \gamma, \quad \gamma\alpha\beta\gamma = \gamma\beta\alpha\beta, \quad \gamma\beta\gamma = 0 \end{aligned} \right\}$$

For any group G the semigroup \mathcal{T} acts on $G \times G$ as follows:

$$\begin{aligned} \alpha : (x, y) &\longrightarrow (xy, y^{-1}) \\ \beta : (x, y) &\longrightarrow (y^{-1}, x^{-1}) \\ \gamma : (x, y) &\longrightarrow (x, 1) \end{aligned}$$

Thus, $G \times G$ turns into \mathcal{T} -set and Theorem 5 takes the form:

Corollary 6 *An idempotent mapping $\sigma : G \times G \rightarrow K$ such that $\sigma(1, 1) = 1$ is a factor set if and only if $\text{supp } \sigma = \{(x, y) \mid \sigma(x, y) \neq 0\}$ is a \mathcal{T} -subset in $G \times G$.*

Now from Corollaries 5 and 6 we get:

Theorem 13 *The Schur multiplier is a semilattice of Abelian groups*

$$Pm(G) = \bigcup_{X \in C(G)} Pm_X(G), \quad PM(G) = \bigcup_{X \in C(G)} PM_X(G),$$

where $C(G)$ is a semilattice of \mathcal{T} -subsets in $G \times G$ with respect to intersection.

Let me say a few words about the semigroup \mathcal{T} . It plays a remarkable role: for any group G it gives a description of idempotent factor sets. Since each PLR is a PPR with an idempotent factor set, we obtain, in particular, some classification of all PLR's of a group G . So \mathcal{T} merits to be considered more thoroughly. Here are more details on its structure.

First of all, its order is 25. The elements α and β generate in \mathcal{T} a subgroup $H = \langle \alpha, \beta \rangle$, isomorphic to the symmetric group S_3 . The complement $U = \mathcal{T} \setminus H$ is an ideal.

One can prove that U is a completely 0-simple semigroup. In the standard notation of Theory of Semigroups [7] it can be written as $U = M^0(D; I, \Lambda; P)$, where $I = \Lambda = \{1, 2, 3\}$, D is a group of the order 2 and P is a (3×3) -sandwich-matrix,

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

7 Cohomology of small categories

In further study of properties of 0-cohomology some difficulties arise because, as I mentioned already, 0-cohomology is not a derived functor in the Abelian category where it is built (see Ex. 3).

So a question appears: is it possible to extend the category of 0-modules so that 0-cohomology becomes a derived functor? One of the useful ways is to pass to bimodules. However, in our situation it does not help as one can see from the following example.

We call an Abelian group A by a 0-bimodule over S if A is right and left 0-module, and besides, $(sa)t = s(at)$ for any $s, t \in S \setminus 0$, $a \in A$. 0-Cohomology of S with values in the category of 0-bimodules are defined similarly to 0-cohomology on 0-modules. Denote it by $HH_0^n(S, A)$.

Example 5 Let $S = \{u, v, w, 0\}$ be a commutative semigroup with multiplication $u^2 = v^2 = uv = w$, $uw = vw = 0$, M a 0-bimodule over S . Then $HH_0^2(S, M) \neq 0$ for $M \neq 0$.

As in Sec. 2, this example shows that in the category of 0-bimodules the cohomology functor HH_0^n is not a derived functor. This is the reason why we use the category $\mathcal{Nat}S$ (which is defined below). Our construction is a generalization of the theory of cohomology for small categories from [2].

As in Sec. 4, we suppose for simplicity that S is a monoid with a zero. Call by the *category of factorizations in S* the category $\mathcal{Fac}S$ whose objects are all elements from $S \setminus 0$, and the set of morphisms $\text{Mor}(a, b)$ consists of all triples (α, a, β) ($\alpha, \beta \in S$) such that $\alpha a \beta = b$ (we will denote (α, a, β) by (α, β) if this does not lead to confusion). The composition of morphisms is defined by the rule $(\alpha', \beta')(\alpha, \beta) = (\alpha' \alpha, \beta \beta')$; hence we have $(\alpha, \beta) = (\alpha, 1)(1, \beta) = (1, \beta)(\alpha, 1)$.

A *natural system on S* is a functor $\mathbf{D} : \mathcal{Fac}S \rightarrow \mathcal{Ab}$. The category $\mathcal{Nat}S = \mathcal{Ab}^{\mathcal{Fac}S}$ of such functors is an Abelian category with enough projectives and injectives [13]. Denote the value of \mathbf{D} on an object $a \in \text{Ob}\mathcal{Fac}S$ by \mathbf{D}_a . If we denote $\alpha_* = \mathbf{D}(\alpha, 1)$ and $\beta^* = \mathbf{D}(1, \beta)$ then $\mathbf{D}(\alpha, \beta) = \alpha_* \beta^*$ for any morphism (α, β) .

Example 6 Each 0-module A can be considered as a functor \mathbf{A} from $\mathcal{Nat}S$, defined as follows: $\mathbf{A}_s = A$ for any $s \in S \setminus 0$ and $\alpha_* \beta^* a = \alpha a$ for all $\alpha, \beta \in S$, $a \in A$.

Example 7 Consider a functor \mathbf{Z} which assigns to each object $a \in S \setminus 0$ the infinite cyclic group \mathbf{Z}_a generated by a symbol $[a]$; to each morphism $(\alpha, \beta) : s \rightarrow t$ it assigns a homomorphism of the groups $\mathbf{Z}(\alpha, \beta) : \mathbf{Z}_a \rightarrow \mathbf{Z}_b$ which takes $[a]$ to $[b]$. It is a natural system, which is called trivial.

For a given natural number n denote by $Ner_n S$ the set of all n -tuples (a_1, \dots, a_n) , $a_i \in S$, such that $a_1 \cdots a_n \neq 0$ (a *nerve* of S). For $n = 0$ we set $Ner_0 S = \{1\}$. A mapping, defined on the nerve and assigning to each $a = (a_1, \dots, a_n)$ an element from $\mathbf{D}_{a_1 \cdots a_n}$, is called an *n -dimensional cochain*. The set of all n -dimensional cochains is an Abelian group $C^n(S, \mathbf{D})$ with respect to the pointwise addition. Set $C^0(S, \mathbf{D}) = \mathbf{D}_1$.

Define a *coboundary homomorphism* $\Delta^n : C^n(S, \mathbf{D}) \rightarrow C^{n+1}(S, \mathbf{D})$ for $n \geq 1$ by the formula

$$\begin{aligned} (\Delta^n f)(a_1, \dots, a_{n+1}) &= a_1 * f(a_2, \dots, a_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i f(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1}) + (-1)^{n+1} a_{n+1}^* f(a_1, \dots, a_n). \end{aligned}$$

For $n = 0$ we set $(\Delta^0 f)(x) = x_* f - x^* f$ where $f \in \mathbf{D}_1$, $x \in S \setminus 0$. One can check directly that $\Delta^n \Delta^{n-1} = 0$. Cohomology groups of the complex $\{C^n(S, \mathbf{D}), \Delta^n\}_{n \geq 0}$ are denoted by $H^n(S, \mathbf{D})$.

0-Cohomology of a monoid is a special case of this construction. Namely, $H_0^*(S, A) \cong H^n(S, \mathbf{A})$, where \mathbf{A} is a functor defined in Ex. 6.

Since $\mathcal{N}at S$ has enough projectives and injectives there exist derived functors $\text{Ext}_{\mathcal{N}at S}^n(\mathbf{Z}, _)$.

Theorem 14 [21, 22] *For any monoid S with zero*

$$H^n(S, _) \cong \text{Ext}_{\mathcal{N}at S}^n(\mathbf{Z}, _).$$

To prove this statement a projective resolution for \mathbf{Z} is built in the following way.

For every $n \geq 0$ we define a natural system $\mathbf{B}_n : \mathcal{F}ac S \rightarrow \mathcal{A}b$. Its value on an object $a \in S \setminus 0$ is a free Abelian group $\mathbf{B}_n(a)$ generated by the set of symbols $[a_0, \dots, a_{n+1}]$ such that $a_0 \cdots a_{n+1} = a$. To each morphism (α, β) we assign a homomorphism of groups by the formula

$$\mathbf{B}_n(\alpha, \beta) : [a_0, \dots, a_{n+1}] \rightarrow [\alpha a_0, \dots, a_{n+1} \beta].$$

These functors constitute a chain complex $\{\mathbf{B}_n, \partial_n\}_{n \geq 0}$, where natural transformations $\partial_n : \mathbf{B}_n \rightarrow \mathbf{B}_{n-1}$ ($n \geq 1$) are given by the homomorphisms

$$(\partial_n)_a : \mathbf{B}_n(a) \rightarrow \mathbf{B}_{n-1}(a),$$

$$(\partial_n)_a[a_0, \dots, a_{n+1}] = \sum_{i=0}^n (-1)^i [a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}].$$

The natural systems \mathbf{B}_n are projective objects in $\mathcal{Nat}S$ and the complex $\{\mathbf{B}_n, \partial_n\}_{n \geq 0}$ is a projective resolution of the natural system \mathbf{Z} .

Now one can establish an isomorphism between the complexes

$$\{C^n(S, \mathbf{D}), \Delta^n\}_{n \geq 0} \quad \text{and} \quad \{\text{Hom}_{\mathcal{Nat}S}(\mathbf{B}_n, \mathbf{D}), \partial^n\}_{n \geq 0}.$$

Our construction differs from Baues cohomology theory for monoids [2] in the initial stage only. Namely, in [2] a monoid S is regarded as a category with a single object. At the same time the Baues category of factorizations in S is equal to $\mathcal{Fac}S^0$ where S^0 is a semigroup with a joined zero. Therefore the Baues cohomology groups of S and the cohomology groups of S^0 in our sense are the same. However if S possesses a zero element then the category $\mathcal{Fac}S$ and Baues one are not equivalent and we obtain the different cohomology groups. The construction of this section is a generalization simultaneously both Baues' and 0-cohomology.

In conclusion I give an example of using obtained results.

It is well-known that in many algebraic theories cohomological dimension of free objects is 1. In the category of monoids with zero a free object is a free monoid with a joined zero. However in this category the class of objects having cohomological dimension 1 is essentially greater.

Call every quotient monoid of a free monoid by its ideal 0-free. Free monoids with a joined zero are also considered as 0-free. Let S be a semigroup with a zero. We call the least n such that $H_0^{n+1}(S, A) = 0$ for any 0-module A , by 0-cohomological dimension (0-cd S) of S .

Theorem 15 0-cd $M \leq 1$ for any 0-free monoid M .

From this theorem it follows an interesting

Corollary 7 Any projective representation of a 0-free monoid is linearized (i. e. is equivalent to a linear one).

In connection with Theorem 15 a question arises, an answer to which is unknown to me: is a 0-cancellative monoid (Sec. 6), having 0-cohomological dimension 1, 0-free?

8 Concluding remarks

General properties of 0-cohomology are studied weakly. Here the same difficulties (and even greater) appear as for EM-cohomology of semigroups.

Certain hopes are given by Theorem 14, showing that 0-cohomology can be continued to a derived functor. However this continuation turns out to be too vast. So it remains actual to find a category, smaller than $\mathcal{F}ac$, in which 0-cohomology would be a derived functor.

0-Cohomology appear in other problems too. In an article by Clark [6] it was applied to semigroups of matrix units and algebras generated by them. Similar situation often occurs in Ring Theory. A quotient algebra of a semigroup algebra by its ideal, generated by the zero of the semigroup, is called a *contracted* one (in other words, the zero of the semigroup is identified with the zero of the algebra). For instance, the well-known theorem of Bautista–Gabriel–Roiter–Salmeron [3] confirms that every algebra of the final type is contracted semigroup one.

A natural question arises: how the Hochschild cohomology of contracted algebras is connected with the cohomology of semigroups generating them? Since it is supposed that the semigroup contains a zero, then, of course, for study of this question it is necessary to use 0-cohomology. Such approach could be useful for incidence algebras of simplicial complexes as well (cf. [11]).

In conclusion let me mention a generalization of 0-cohomology. In a semigroup S (not necessary containing 0) let us fix certain generating subset $W \subset S$ instead of $S \setminus 0$ and call mappings $W \rightarrow A$ by *1-dimensional W -coboundaries*. Using this one can construct certain partial n -place mappings $S \times \dots \times S \rightarrow A$ (and call them *n -dimensional W -cochains*) so that coboundary homomorphisms ∂^n would be well defined. I have called the obtained objects by *partial cohomology* (they have no concern with partial representations from Sec.6) and considered it in [34] and [35]. Partial cohomology turned out to be useful for calculation of EM-cohomology (as in Sec.3), however I have not meet other applications of them.

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